

# On $N$ -wave Type Systems and Their Gauge Equivalent

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**Abstract.** The class of nonlinear evolution equations (NLEE) – gauge equivalent to the  $N$ -wave equations related to the simple Lie algebra  $\mathfrak{g}$  are derived and analyzed. They are written in terms of  $\mathcal{S}(x, t) \in \mathfrak{g}$  satisfying  $r = \text{rank } \mathfrak{g}$  nonlinear constraints. The corresponding Lax pairs and the time evolution of the scattering data are found. The Zakharov–Shabat dressing method is appropriately modified to construct their soliton solutions.

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## 1 Introduction and Preliminaries

It is well known [1, 2] that the Lax representation (1)

$$[L(\lambda), M(\lambda)] = 0 \quad (1)$$

is invariant under the group of gauge transformations.

One of the first nontrivial examples of gauge equivalent nonlinear evolution equations (NLEE) is provided by the nonlinear Schrödinger equation (NLSE) [3, 1, 2]:

$$iu_t + u_{xx} + 2|u|^2 u(x, t) = 0, \quad (2)$$

and the Heisenberg ferromagnet equation (HFE):

$$iS_t^{(0)} = \frac{1}{2}[S^{(0)}(x, t), S_{xx}^{(0)}], \quad (3)$$

$$S^{(0)}(x, t) = g^{(0)-1} \sigma_3 g^{(0)}(x, t);$$

obviously  $(S^{(0)})^2 = \mathbb{1}$ . The equivalence between (2) and (3) is based on the fact that  $g^{(0)}(x, t)$  is determined by  $u(x, t)$  through (see [2]):

$$i \frac{dg^{(0)}}{dx} + q^{(0)}(x, t) g^{(0)}(x, t) = 0, \quad (4)$$

$$q^{(0)}(x, t) = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \quad \lim_{x \rightarrow \infty} g^{(0)}(x, t) = \mathbb{1}. \quad (5)$$

Both equations are infinite dimensional completely integrable Hamiltonian systems. The phase space  $\mathcal{M}_{\text{NLSE}}$  is the linear space of all off-diagonal matrices  $q^{(0)}(x, t)$  tending fast enough to zero for  $x \rightarrow \pm\infty$ . A hierarchy of

pair-wise compatible symplectic structures on  $\mathcal{M}_{\text{NLSE}}$  is provided by the 2-forms:

$$\Omega_{\text{NLSE}}^{(k)} = \frac{i}{4} \int_{-\infty}^{\infty} dx \text{tr} \left( \delta q^{(0)} \wedge \Lambda^k [\sigma_3, \delta q^{(0)}(x, t)] \right). \quad (6)$$

The phase space  $\mathcal{M}_{\text{HFE}}$  of the HFE is the manifold of all  $S^{(0)}(x, t)$  determined by the second relation in (3). The family of compatible 2-forms is:

$$\tilde{\Omega}_{\text{HFE}}^{(k)} = \frac{i}{4} \int_{-\infty}^{\infty} dx \text{tr} \left( \delta S^{(0)} \wedge \tilde{\Lambda}^k [S^{(0)}, \delta S^{(0)}(x, t)] \right). \quad (7)$$

By  $\Lambda$  and  $\tilde{\Lambda}$  we have denoted the recursion operator of the NLS type equations and its gauge equivalent [4]. The spectral theory of these two operators underlie all the fundamental properties of these two classes of gauge equivalent NLEE, for details see [4]. Note that the gauge transformation relates nontrivially the symplectic structures, i.e.  $\Omega_{\text{NLSE}}^{(k)} \simeq \tilde{\Omega}_{\text{HFE}}^{(k+2)}$  [5, 4].

The NLSE is solvable by the inverse scattering method applied to the Zakharov–Shabat system. It can be generalized to any simple Lie algebra  $\mathfrak{g}$  of rank  $r > 1$  by [6, 7, 8, 9, 10, 11]:

$$L(\lambda)\psi \equiv \left( i \frac{d}{dx} + q(x, t) - \lambda J \right) \psi(x, t, \lambda) = 0, \quad (8)$$

where  $q(x, t)$  and  $J$  are elements  $\mathfrak{g}$ . A natural choice of the gauge in (8) consists in choosing  $J$  to be a constant regular element of the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Next by a gauge transformation  $L(\lambda) \rightarrow g_0^{-1} L(\lambda) g_0$  where  $g_0(x, t) \in \mathfrak{G}$  commutes with  $J$  we can eliminate the diagonal elements of  $q(x, t)$  and cast it in the form  $q(x, t) = [J, Q(x, t)]$ . Since  $J$  is a regular element of  $\mathfrak{h}$  then  $q(x, t) \in \mathfrak{g} \setminus \mathfrak{h}$  is determined

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by  $|\Delta|$  independent coefficient functions as follows:

$$Q(x, t) = \sum_{\alpha \in \Delta^+} (q_\alpha(x, t)E_\alpha + p_\alpha(x, t)E_{-\alpha}), \quad (9)$$

where  $E_{\pm\alpha}$  are the root vectors of  $\mathfrak{g}$  and  $\Delta_+$  is the set of positive roots of  $\mathfrak{g}$ ,  $\Delta = \Delta^+ \cup (-\Delta^+)$ . The root  $\alpha$  is positive if  $\alpha(J) > 0$ ; by  $E_\alpha$ ,  $\alpha \in \Delta$  and  $H_i$ ,  $i = 1, \dots, r$  we denote the Cartan–Weyl basis of  $\mathfrak{g}$  with the standard commutation relations, see [12]. This choice of the gauge of  $L(\lambda)$  allows one to study the  $N$ -wave type equations:

$$i[J, Q_t] - i[I, Q_x] + [[I, Q], [J, Q]] = 0. \quad (10)$$

Indeed eq. (10) allows Lax representation with the pair of Lax operators  $L(\lambda)$  (8) and  $M(\lambda)$ :

$$M(\lambda)\psi \equiv \left(i \frac{d}{dt} + [I, Q(x, t)] - \lambda I\right) \psi(x, t, \lambda) = 0, \quad (11)$$

where  $I$  is also a constant regular element of  $I \in \mathfrak{h}$ .

There is a second canonical way to fix up the gauge of the Lax operator known as the pole gauge [13]:

$$\tilde{L}\tilde{\psi}(x, t, \lambda) \equiv \left(i \frac{d}{dx} - \lambda \mathcal{S}(x, t)\right) \tilde{\psi}(x, t, \lambda) = 0, \quad (12)$$

where  $\tilde{\psi}(x, t, \lambda) = g^{-1}(x, t)\psi(x, t, \lambda)$ ,

$$\mathcal{S}(x, t) = \text{Ad}_g \cdot J \equiv g^{-1}(x, t)Jg(x, t). \quad (13)$$

The gauge transformation which takes  $L(\lambda)$  to  $\tilde{L}(\lambda) = g^{-1}L(\lambda)g(x, t)$  is performed with the Jost solution of (8) taken at  $\lambda = 0$ , i.e.  $g(x, t) \in \mathfrak{G}$  and

$$i \frac{dg}{dx} + q(x, t)g(x, t) = 0, \quad \lim_{x \rightarrow \infty} g(x, t) = \mathbb{1}. \quad (14)$$

Applying the gauge transformation to  $M(\lambda)$  we get:

$$\tilde{M}\tilde{\psi}(x, t, \lambda) \equiv \left(i \frac{d}{dt} - \lambda f(\mathcal{S})\right) \tilde{\psi}(x, t, \lambda) = 0, \quad (15)$$

where  $f(\mathcal{S})$  is a function (in fact, a polynomial) to be determined below. Indeed only this choice of  $\tilde{M}(\lambda)$  ensures the vanishing of the  $\lambda^2$ -term in the zero-curvature condition  $[\tilde{L}, \tilde{M}] = 0$ ; the terms proportional to  $\lambda$  lead to the following form of the gauge equivalent NLEE:

$$\mathcal{S}_t - \frac{d}{dx}f(\mathcal{S}) = 0, \quad (16)$$

While the  $N$ -wave type equations are well known their gauge equivalent ones to the best of our knowledge have not been derived yet. One of the difficulties in doing this is the necessity to express all factors in terms of  $\mathcal{S}$  only.

The gauge equivalent operators  $L(\lambda)$  and  $\tilde{L}(\lambda)$  have equivalent spectral properties and spectral data. This fact allows one to prove that the classes of NLEE related to  $L(\lambda)$  and  $\tilde{L}(\lambda)$  are also equivalent.

In Section 2 we construct the NLEE gauge equivalent to the  $N$ -wave equations (10) extending the results in [4].

Namely we calculate the functions  $f(\mathcal{S})$  for the cases when  $\mathfrak{g}$  belongs to the classical series of simple Lie algebras. In Section 3 we briefly describe the interrelations between the scattering data of  $L(\lambda)$  and  $\tilde{L}(\lambda)$  and outline some of their reductions. We also reformulate the Riemann–Hilbert problem (RHP) for the gauge equivalent systems and describe the time evolution of the scattering data. In section 4 we extend the Zakharov–Shabat dressing method [7, 9, 13] for the gauge equivalent systems [4] and provide the general form of the 1-soliton solution of these system. These results are demonstrated on an example on the orthogonal Lie algebra  $\mathbf{B}_2 \simeq so(5)$  in section 5. This and other particular cases of Eqs. (16) describe isoparametric hypersurfaces [14].

## 2 General form of the gauge equivalent systems

It is natural that  $f(\mathcal{S}) = g^{-1}(x, t)Ig(x, t)$ , i.e., it is uniquely determined by  $I$ . Both  $J$  and  $I$  belong to the Cartan subalgebra  $\mathfrak{h}$  so they have common set of eigenspaces.

The derivation of the corresponding functions  $f(\mathcal{S})$  is different for  $\mathbf{A}_r$  and for  $\mathbf{B}_r$ ,  $\mathbf{C}_r$ ,  $\mathbf{D}_r$  series. Let first  $\mathfrak{g} \simeq \mathbf{A}_r = sl(n)$  with  $n = r + 1$ . Then

$$J = \text{diag}(J_1, \dots, J_n), \quad I = \text{diag}(I_1, \dots, I_n),$$

and the only constraint on the eigenvalues  $J_k$  and  $I_k$  is  $\text{tr } J = \text{tr } I = 0$ . The projectors on the common eigensubspaces of  $J$  and  $I$  are given by:

$$\pi_k(J) = \prod_{s \neq k} \frac{J - J_s}{J_k - J_s} = \text{diag}(0, \dots, 0, \underset{k}{1}, 0, \dots, 0). \quad (17)$$

Next we note that

$$I = \sum_{k=1}^n I_k \pi_k(J). \quad (18)$$

In order to derive  $f(\mathcal{S})$  for  $\mathfrak{g} \simeq sl(n)$  we need to apply the gauge transformation to (18) with the result:

$$f(\mathcal{S}) = \sum_{k=1}^n I_k \pi_k(\mathcal{S}), \quad (19)$$

i.e.,  $f(\mathcal{S})$  is a polynomial of order  $n - 1$ . Obviously  $\mathcal{S}$  is restricted by:

$$\prod_{k=1}^n (\mathcal{S} - J_k) = 0, \quad \text{tr } \mathcal{S}^k = \text{tr } J^k, \quad (20)$$

for  $k = 2, \dots, n$ .

Let us now assume that  $\mathfrak{g}$  is an orthogonal or symplectic algebra. In the typical representation we can introduce the Cartan generators  $H_{e_k}$  which are dual to the orthogonal basic vectors  $e_k$  in the root space. Each  $H_{e_k}$  has only

two non-vanishing eigenvalues equal to 1 and  $-1$  respectively. Then we put:

$$J = \sum_{k=1}^r J_k H_{e_k}, \quad I = \sum_{k=1}^r I_k H_{e_k}.$$

Obviously the odd powers of  $H_{e_k}$  also belong to  $\mathfrak{g}$  while the even powers do not. The projectors  $f_k(J)$  onto  $H_{e_k}$  then can be written down as:

$$f_k(J) = \frac{J}{J_k} \prod_{s \neq k} \frac{J^2 - J_s^2}{J_k^2 - J_s^2} = H_{e_k} \in \mathfrak{h}. \quad (21)$$

Therefore

$$I = \sum_{k=1}^r I_k f_k(J), \quad (22)$$

and applying the gauge transformation we get:

$$f(\mathcal{S}) \equiv g^{-1}(x, t) I g(x, t) = \sum_{k=1}^r I_k f_k(\mathcal{S}). \quad (23)$$

Then the equation gauge equivalent to (8) is given by (16) with  $f(\mathcal{S})$  determined by (23).

In addition  $\mathcal{S}(x, t)$  satisfies a set of nonlinear constraints; one of them is the characteristic equation:

$$\mathcal{S}^{\kappa_0} \prod_{k=1}^r (\mathcal{S}^2 - J_k^2) = 0, \quad (24)$$

where  $\kappa_0 = 0$  if  $\mathfrak{g} \simeq C_r$  or  $D_r$  and  $\kappa_0 = 1$ , if  $\mathfrak{g} \simeq B_r$ . To construct the others we use the typical representation of  $\mathfrak{g}$ . In this settings we see that all even powers of  $H_{e_k}$  have trace equal to 2. Thus we have:

$$\text{tr}(J^{2k}) \equiv 2 \sum_{p=1}^r J_p^{2k} = \text{tr}(\mathcal{S})^{2k}, \quad (25)$$

for  $k = 1, \dots, r$ . The conditions (25) are precisely  $r$  independent algebraic constraints on  $\mathcal{S}$ . Solving for them we conclude that the number of independent coefficients in  $\mathcal{S}$  is equal to the number of roots  $|\Delta|$  of  $\mathfrak{g}$ .

Both classes of NLEE possess hierarchies of Hamiltonian structures. The phase space  $\mathcal{M}_{N-w}$  of the  $N$ -wave equations is the linear space of off-diagonal matrices  $q(x, t)$ ; the hierarchy of symplectic structures is given by:

$$\Omega_{N-w}^{(k)} = i \int_{-\infty}^{\infty} dx \text{tr} (\delta q \wedge \Lambda^k [J, \delta q(x, t)]). \quad (26)$$

The phase space  $\mathcal{M}_{\mathcal{S}}$  of their gauge equivalent equations (16) is the nonlinear manifold of all  $\mathcal{S}(x, t)$  satisfying equations (24), (25). The family of compatible 2-forms is:

$$\tilde{\Omega}_{\mathcal{S}}^{(k)} = i \int_{-\infty}^{\infty} dx \text{tr} (\delta \mathcal{S} \wedge \tilde{\Lambda}^k [\mathcal{S}, \delta \mathcal{S}(x, t)]). \quad (27)$$

Here  $\Lambda$  and  $\tilde{\Lambda}$  are the recursion operator of the  $N$ -wave type equations (see [4]) and its gauge equivalent:  $\tilde{\Lambda} = g^{-1} \Lambda g(x, t)$ .

### 3 Fundamental analytic solutions (FAS) and scattering data for gauge equivalent systems

The direct scattering problem for the Lax operator (8) is based on the Jost solutions:

$$\lim_{x \rightarrow \infty} \psi(x, \lambda) e^{i\lambda J x} = \mathbb{1}, \quad \lim_{x \rightarrow -\infty} \phi(x, \lambda) e^{i\lambda J x} = \mathbb{1}, \quad (28)$$

and the scattering matrix:

$$T(\lambda) = (\psi(x, \lambda))^{-1} \phi(x, \lambda). \quad (29)$$

The FAS  $\xi^{\pm}(x, \lambda)$  of  $L(\lambda)$  are analytic functions of  $\lambda$  for  $\lambda \gtrless 0$  and are related to the Jost solutions by [11]

$$\xi^{\pm}(x, \lambda) = \phi(x, \lambda) S^{\pm}(\lambda) = \psi(x, \lambda) T^{\mp}(\lambda) D^{\pm}(\lambda), \quad (30)$$

where  $T^{\pm}(\lambda)$ ,  $S^{\pm}(\lambda)$  and  $D^{\pm}(\lambda)$  are the factors of the Gauss decomposition of the scattering matrix:

$$T(\lambda) = T^{-}(\lambda) D^{+}(\lambda) \hat{S}^{+}(\lambda) = T^{+}(\lambda) D^{-}(\lambda) \hat{S}^{-}(\lambda). \quad (31)$$

Here  $\hat{S} \equiv S^{-1}$ , the superscripts  $+$  (resp.  $-$ ) in  $T^{\pm}(\lambda)$  and  $S^{\pm}(\lambda)$  mean upper- (resp. lower-) triangularity. The diagonal factors  $D^{\pm}(\lambda)$  are analytic functions of  $\lambda$  for  $\text{Im } \lambda > 0$  and  $\text{Im } \lambda < 0$  respectively.

On the real axis  $\xi^{+}(x, \lambda)$  and  $\xi^{-}(x, \lambda)$  are related by

$$\xi^{+}(x, \lambda) = \xi^{-}(x, \lambda) G_0(\lambda), \quad G_0(\lambda) = \hat{S}^{-}(\lambda) S^{+}(\lambda), \quad (32)$$

and the function  $G_0(\lambda)$  can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of (8) [15, 11].

If the potential  $q(x, t)$  of  $L(\lambda)$  (8) satisfies equation (10) then  $S^{\pm}(\lambda)$  and  $T^{\pm}(\lambda)$  satisfy the linear equation:

$$i \frac{dS^{\pm}}{dt} - \lambda [I, S^{\pm}] = 0 \quad i \frac{dT^{\pm}}{dt} - \lambda [I, T^{\pm}] = 0, \quad (33)$$

while the functions  $D^{\pm}(\lambda)$  are time-independent. In other words  $D^{\pm}(\lambda)$  can be considered as the generating functions of the integrals of motion of (10).

In order to determine the scattering data for the gauge equivalent equations we need the FAS for these systems:

$$\tilde{\xi}^{\pm}(x, \lambda) = g^{-1}(x, t) \xi^{\pm}(x, \lambda) g_{-}, \quad (34)$$

where  $g_{-} = \lim_{x \rightarrow -\infty} g(x, t) = \hat{T}(0)$ . In order to ensure that the functions  $\tilde{\xi}^{\pm}(x, \lambda)$  are analytic with respect to  $\lambda$  the scattering matrix  $T(0)$  at  $\lambda = 0$  must belong to the corresponding Cartan subgroup  $\mathfrak{h}$ . Then Equation (34) provide the FAS of  $\tilde{L}$ . We calculate their asymptotics for  $x \rightarrow \pm\infty$  and establish the relations between the scattering matrices of the two systems:

$$\lim_{x \rightarrow -\infty} \tilde{\xi}^{+}(x, \lambda) = T(0) S^{+}(\lambda) \hat{T}(0) \quad (35)$$

$$\lim_{x \rightarrow \infty} \tilde{\xi}^{+}(x, \lambda) = e^{-i\lambda J x} T^{-}(\lambda) D^{+}(\lambda) \hat{T}(0) \quad (36)$$

with the result:

$$\tilde{T}(\lambda) = T(\lambda) \hat{T}(0). \quad (37)$$

Obviously  $\tilde{T}(0) = \mathbb{1}$ . The factors in the corresponding Gauss decompositions are related by:

$$\begin{aligned} \tilde{S}^\pm(\lambda) &= T(0)S^\pm(\lambda)\hat{T}(0), & \tilde{T}^\pm(\lambda) &= T^\pm(\lambda) \\ \tilde{D}^\pm(\lambda) &= D^\pm(\lambda)\hat{T}(0). \end{aligned} \quad (38)$$

On the real axis  $\tilde{\xi}^+(x, \lambda)$  and  $\tilde{\xi}^-(x, \lambda)$  are related by:

$$\tilde{\xi}^+(x, \lambda) = \tilde{\xi}^-(x, \lambda)\tilde{G}_0(\lambda), \quad \tilde{G}_0(\lambda) = \hat{S}^-(\lambda)\tilde{S}^+(\lambda) \quad (39)$$

with the normalization condition  $\tilde{\xi}(x, 0) = \mathbb{1}$ ; again  $\tilde{G}_0(\lambda)$  can be considered as a minimal set of scattering data.

The numerous  $\mathbb{Z}_2$ -reductions that have been recently classified for the  $N$ -wave equations [16,17] using the reduction group introduced by Mikhailov [18]. They can easily be reformulated for the gauge equivalent systems. Here we write down only two of them:

$$\begin{aligned} 1) \quad & \mathcal{S}^\dagger(x, t) = K\mathcal{S}(x, t)K^{-1}, \quad K \in \mathfrak{H}, \quad K^2 = \mathbb{1}, \\ 2) \quad & \mathcal{S}(x, t) = \mathcal{S}(x, t)^T. \end{aligned} \quad (40)$$

Obviously each of the constraints 1) and 2) are compatible with equation (16) and diminishes the number of independent coefficients by a factor of 2.

## 4 Dressing factors and 1-soliton solutions

The main idea of the dressing method is starting from a FAS  $\tilde{\xi}_{(0)}^\pm(x, \lambda)$  of  $\tilde{L}$  with potential  $\mathcal{S}_{(0)}$  to construct a new singular solution  $\tilde{\xi}_{(1)}^\pm(x, \lambda)$  of the RHP (39) with singularities located at prescribed positions  $\lambda_1^\pm$ . Then the new solutions  $\tilde{\xi}_{(1)}^\pm(x, \lambda)$  will correspond to a potential  $\mathcal{S}_{(1)}$  of  $\tilde{L}$  with two discrete eigenvalues  $\lambda_1^\pm$ . It is related to the regular one by the dressing factors  $\tilde{u}(x, \lambda)$ :

$$\begin{aligned} \tilde{\xi}_{(1)}^\pm(x, \lambda) &= \tilde{u}(x, \lambda)\tilde{\xi}_{(0)}^\pm(x, \lambda)\tilde{u}_-^{-1}(\lambda), \\ \tilde{u}_-(\lambda) &= \lim_{x \rightarrow -\infty} \tilde{u}(x, \lambda), \end{aligned} \quad (41)$$

and the dressing factors for the gauge equivalent equations  $\tilde{u}(x, \lambda)$  are related to  $u(x, \lambda)$  by

$$\tilde{u}(x, \lambda) = g_{(0)}^{-1}(x, t)u^{-1}(x, \lambda = 0)u(x, \lambda)g_{(0)}. \quad (42)$$

If  $\mathfrak{g} \simeq \mathbf{A}_r$  then the gauge equivalent dressing factors are

$$\begin{aligned} \tilde{u}(x, \lambda) &= \mathbb{1} + \left( \frac{c_1(\lambda)}{c_1(0)} - 1 \right) P_1, \quad c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-} \\ P_1(x) &= \frac{|n(x)\rangle\langle m(x)|}{\langle m(x)|n(x)\rangle}, \\ |n(x)\rangle &= \xi_0^+(\lambda_1^+)|n_0\rangle, \quad \langle m(x)| = \langle m_0|\xi_0^-(\lambda_1^-), \end{aligned} \quad (43)$$

where  $|n_0\rangle$  and  $\langle m_0|$  are constant vectors and these dressing factors satisfy the equation:

$$i \frac{d\tilde{u}}{dx} - \lambda \mathcal{S}_{(1)}\tilde{u} + \lambda \tilde{u}\mathcal{S}_{(0)} = 0. \quad (44)$$

If  $\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{D}_r$  the dressing factors take the form [16]:

$$u(x, \lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1 + (c_1^{-1}(\lambda) - 1)P_{-1} \quad (45)$$

$$\tilde{u}(x, \lambda) = \mathbb{1} + \left( \frac{c_1(\lambda)}{c_1(0)} - 1 \right) \tilde{P}_1 + \left( \frac{c_1(0)}{c_1(\lambda)} - 1 \right) \tilde{P}_{-1} \quad (46)$$

where  $P_{-1}(x) = SP_1^T(x)S^{-1}$ ,  $P_1(x)$  is the rank 1 projector (43),  $\tilde{P}_{\pm 1} = g_{(0)}^{-1}P_{\pm 1}g_{(0)}(x, t)$ . If  $\mathfrak{g} \simeq B_r$  then  $N = 2r + 1$ ,

$$S = \sum_{k=1}^r (-1)^{k+1} (E_{k\bar{k}} + E_{\bar{k}k}) + (-1)^r E_{r+1, r+1}; \quad (47)$$

$\bar{k} = N - k + 1$ ,  $(E_{km})_{il} = \delta_{ik}\delta_{ml}$ ; if  $\mathfrak{g} \simeq D_r$  then  $N = 2r$

$$S = \sum_{k=1}^r (-1)^{k+1} (E_{k\bar{k}} + E_{\bar{k}k}), \quad (48)$$

If the dressing factors of the gauge equivalent equations satisfy (44) then the projectors  $\tilde{P}_{\pm 1}$  satisfy the equations:

$$\begin{aligned} i \frac{d\tilde{P}_1}{dx} + \lambda_1^- \tilde{P}_1 \mathcal{S}_{(0)} - \lambda_1^- \mathcal{S}_{(1)} \tilde{P}_1 &= 0, \\ i \frac{d\tilde{P}_{-1}}{dx} + \lambda_1^+ \tilde{P}_{-1} \mathcal{S}_{(0)} - \lambda_1^+ \mathcal{S}_{(1)} \tilde{P}_{-1} &= 0, \end{aligned} \quad (49)$$

and the "dressed" potential can be obtained by:

$$\mathcal{S}_{(1)} = \mathcal{S}_{(0)} + i \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ \lambda_1^-} \frac{d}{dx} (\tilde{P}_1(x) - \tilde{P}_{-1}(x)). \quad (50)$$

The dressing factors can be written in the form:

$$\tilde{u}(x, \lambda) = \exp \left[ \ln \left( \frac{c_1(\lambda)}{c_1(0)} \right) \tilde{p}(x) \right], \quad (51)$$

where  $\tilde{p}(x) = \tilde{P}_1 - \tilde{P}_{-1} \in \mathfrak{g}$  and consequently  $\tilde{u}(x, \lambda)$  belongs to the corresponding orthogonal group.

Making use of the explicit form of the projectors  $P_{\pm 1}(x)$  valid for the typical representation of  $\mathbf{B}_r$  we have [16]

$$\begin{aligned} \tilde{p}(x) &= \frac{2}{\langle m|n \rangle} \sum_{k=1}^r \tilde{h}_k(x) H_{e_k} \\ &+ \frac{2}{\langle m|n \rangle} \sum_{\alpha \in \Delta_+} (\tilde{P}_\alpha(x) E_\alpha + \tilde{P}_{-\alpha}(x) E_{-\alpha}), \end{aligned} \quad (52)$$

where we assumed  $\mathcal{S}_{(0)} = J$ ,  $g_{(0)} = \mathbb{1}$ . Thus

$$\begin{aligned} \tilde{h}_k(x, t) &= n_{0,k} m_{0,k} e^{2\nu_1 y_k} - n_{0,\bar{k}} m_{0,\bar{k}} e^{-2\nu_1 y_k}, \\ \langle m|n \rangle &= \sum_{k=1}^r (n_{0,k} m_{0,k} e^{2\nu_1 y_k} + n_{0,\bar{k}} m_{0,\bar{k}} e^{-2\nu_1 y_k}) \\ &+ n_{0,r+1} m_{0,r+1}, \end{aligned} \quad (53)$$

$$\tilde{P}_\alpha = \begin{cases} \tilde{P}_{ks}, & \text{for } \alpha = e_k - e_s \\ \tilde{P}_{\bar{k}\bar{s}}, & \text{for } \alpha = e_k + e_s \\ \tilde{P}_{k,r+1}, & \text{for } \alpha = e_k \end{cases}$$

Here  $1 \leq k, s \leq r$ ,  $\mu_1 = \text{Re } \lambda_1^+$ ,  $\nu_1 = \text{Im } \lambda_1^+$  and

$$\begin{aligned} \tilde{P}_{ks} &= e^{i\mu_1(y_s - y_k)} (n_{0,k} m_{0,s} e^{\nu_1(y_s + y_k)} \\ &\quad - (-1)^{k+s} n_{0,\bar{s}} m_{0,\bar{k}} e^{-\nu_1(y_s + y_k)}), \\ y_k &= J_k x + I_k t, \quad y_{\bar{k}} = -y_k, \quad y_{r+1} = 0. \end{aligned} \quad (54)$$

The corresponding result for the  $\mathbf{D}_r$  series is obtained formally if in the above expressions (53), (54) and (54) we put  $n_{0,r+1} = m_{0,r+1} = 0$ . Thus  $\tilde{P}_{k,r+1} = \tilde{P}_{r+1,k} = 0$  and the last term in the right hand side of  $\langle m|n \rangle$  (53) is missing.

The reductions (40) applied to the 1-soliton solution constraint the vectors  $|n_0\rangle$ ,  $\langle m_0|$  and the eigenvalues  $\lambda_1^\pm$ :

$$\begin{aligned} 1) \quad |n_0\rangle &= K|m_0^*\rangle, \quad \lambda_1^- = (\lambda_1^+)^*, \\ 2) \quad |n_0\rangle &= |m_0\rangle, \quad \lambda_1^- = -\lambda_1^+. \end{aligned} \quad (55)$$

The  $N$ -soliton solutions can be obtained by applying successively  $N$  times the dressing procedure.

## 5 Example: $\mathfrak{g} \simeq \mathbf{B}_2$ algebra

This algebra has four positive roots:  $e_1 \pm e_2$ ,  $e_1$  and  $e_2$ . The corresponding 4-wave system has the form:

$$\begin{aligned} i(J_1 - J_2)q_{10,t} - i(I_1 - I_2)q_{10,x} + 2\kappa q_{11}q_{01}^* &= 0, \\ iJ_2q_{01,t} - iI_2q_{01,x} + \kappa(q_{11}^*q_{12} + q_{11}q_{10}^*) &= 0, \\ iJ_1q_{11,t} - iI_1q_{11,x} + \kappa(q_{12}q_{01}^* - q_{10}q_{01}) &= 0, \\ i(J_1 + J_2)q_{12,t} - i(I_1 + I_2)q_{12,x} - 2\kappa q_{11}q_{01} &= 0. \end{aligned} \quad (56)$$

where  $\kappa = J_1I_2 - J_2I_1$  and the subscripts 10, 01, 11 and 12 refer to the roots  $e_1 - e_2$ ,  $e_1$ ,  $e_2$  and  $e_1 + e_2$  respectively. This system has applications in nonlinear optics [9, 16] and in differential geometry [14]. Its gauge equivalent is:

$$\begin{aligned} \mathcal{S}t - f_1\mathcal{S}x - f_3(\mathcal{S}^3)_x &= 0, \\ f_1 &= \frac{I_2J_1^3 - I_1J_2^3}{J_1J_2(J_1^2 - J_2^2)}, \quad f_3 = \frac{I_1J_2 - I_2J_1}{J_1J_2(J_1^2 - J_2^2)}, \end{aligned} \quad (57)$$

where the  $5 \times 5$  matrix  $\mathcal{S}$  is constrained by:

$$\begin{aligned} \text{tr } \mathcal{S}^2 &= 2(J_1^2 + J_2^2), \quad \text{tr } \mathcal{S}^4 = 2(J_1^4 + J_2^4), \\ \mathcal{S}(\mathcal{S}^2 - J_1^2)(\mathcal{S}^2 - J_2^2) &= 0. \end{aligned} \quad (58)$$

We write down the 1-soliton solution for a special choice

$$\begin{aligned} n_{0,1} &= 1, \quad n_{0,2} = \rho, \quad n_{0,3} = \sqrt{2(\rho^2 - 1)}, \\ n_{0,k} &= n_{0,\bar{k}}, \quad m_{0,k} = n_{0,k}. \end{aligned} \quad (59)$$

of the soliton parameters with  $\rho \geq 1$  and real. The choice (59) satisfies (55) with  $K = \mathbb{1}$ . Inserting it into the general formulae (52)–(54) we get  $\tilde{P}_{-\alpha} = \tilde{P}_\alpha^*$  with:

$$\begin{aligned} \langle m|n \rangle &= 2(\sinh^2 \nu_1 y_1 + \rho^2 \cosh^2 \nu_1 y_2), \\ \tilde{h}_1 &= \sinh 2\nu_1 y_1, \quad \tilde{h}_2 = \rho^2 \sinh 2\nu_1 y_2, \\ \tilde{P}_{e_1 \pm e_2} &= \rho e^{-i\mu_1(y_1 \pm y_2)} \cosh \nu_1(y_1 \mp y_2), \\ \tilde{P}_{e_k} &= \sqrt{2(\rho^2 - 1)} e^{-i\mu_1 y_k} \sinh \nu_1 y_k, \quad k = 1, 2. \end{aligned} \quad (60)$$

If  $\rho = 1$  we get a 1-soliton solution associated with the  $D_2 \simeq A_1 \oplus A_1$  subalgebra;  $\rho = 0$  gives a 1-soliton solution for the  $so(3)$  subalgebra of  $B_2$ . In both subcases the subsets of roots ( $\{\pm e_1 \pm e_2\}$  and  $\{\pm e_1, \pm e_2\}$  resp.) for which  $\tilde{P}_\alpha \neq 0$  contain only roots with the same length.

## 6 Discussion

We derived the explicit form of the NLEE gauge equivalent to the  $N$ -wave equations related to the classical simple Lie algebras  $\mathfrak{g}$ . These equations are Hamiltonian ones. Their phase space  $\tilde{\mathcal{M}}$  is a nonlinear one, since  $\mathcal{S}$  is restricted by the nonlinear constraints (24), (25).

The gauge covariant formulation of the spectral decompositions of the recursion operators  $\Lambda$  and its gauge equivalent  $\tilde{\Lambda}$  allows one to expect that the hierarchies of symplectic structures will satisfy in analogy with the NLSE-HFE case the relation  $\Omega_{N-w}^{(k)} \simeq \tilde{\Omega}_S^{(k+2)}$ .

Another open problem is the study of the  $\mathbb{Z}_2$ -reductions of (16) along the ideas outlined in [16, 17].

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